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The spread of unicyclic graphs with given size of maximum matchings

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The spread $s(G)$ of a graph G is defined as $s(G) = \max_{i,j} |\lambda_i - \lambda_j|$, where the maximum is taken over all pairs of eigenvalues of G. Let $U(n,k)$ denote the set of all unicyclic graphs on n vertices with a maximum matching of cardinality k, and $U^*(n,k)$ the set of triangle-free graphs in $U(n,k)$. In this paper, we determine the graphs with the largest and second largest spectral radius in $U^*(n,k)$, and the graph with the largest spread in $U(n,k)$.

KEY WORDS: spread, unicyclic graph, characteristic polynomial, eigenvalue

AMS subject classification: 05C50, 15A42, 15A36

1. Introduction

All graphs $G = (V, E)$ considered here are finite, undirected and simple. Let G be a graph with n vertices and $A(G)$ the adjacency matrix of G. The characteristic polynomial of $A(G)$ is $\phi(G, \lambda) = \det(\lambda I - A(G))$. The roots $\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)(\lambda_1(G) \geqslant \lambda_2(G) \geqslant \ldots \geqslant \lambda_n(G))$ of $\phi(G, \lambda) = 0$ are called the eigenvalues of G. Since, $A(G)$ is symmetric, all the eigenvalues of G are real.

The spread $s(G)$ of a graph G is defined as $s(G) = \max_{i,j} |\lambda_i - \lambda_j|$, where the maximum is taken over all pairs of eigenvalues of G. The spread of G is also defined as $s(G) = \lambda_1 - \lambda_n$, where λ_1, λ_n are the largest and least eigenvalues of $A(G)$, respectively. There have been some studies on the spread of an arbitrary matrix and a graph (see [9,11,12]).

Let $U(n,k)$ denote the set of all unicyclic graphs on n vertices with a maximum matching of cardinality k, and $U^*(n,k)$ the set of triangle-free graphs

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in $U(n,k)$. For a unicyclic graph G, let $C(G)$ denote the unique cycle of G and $g(G)$ the length of $C(G)$.

Let $S_3^1(n,k)$ denote the graph on n vertices obtained from C_3 by attaching $n-2k+1$ pendant edges and $k-2$ paths of length 2 to a vertex of C_3, and $S_3^2(n,k)$ the graph on n vertices obtained from C_3 by attaching $n-2k+1$ pendant edges and $k-3$ paths of length 2 to a vertex of C_3, and one pendant edge to each of the other two vertices of C_3. Let $S_3^3(n,k)$ denote the graph on n vertices obtained from C_3 by attaching $n-2k$ pendant edges and $k-2$ paths of length 2 to a vertex of C_3, and one pendant edge to one of the other two vertices of C_3 (see figure 1).

Let $S_4^1(n,k)$ denote the graph on n vertices obtained from C_4 by attaching $n-2k+1$ pendant edges and $k-3$ paths of length 2 to one vertex of C_4, and one pendant edge to the adjacent vertex of C_4. Let $S_4^2(n,k)$ denote the graph on n vertices obtained from C_4 by attaching $n-2k$ pendant edges and $k-2$ paths of length 2 to one vertex of C_4. Let $S_4^3(n,k)$ denote the graph obtained from $S_4^1(n-3,k)$ by attaching three pendant edges to the three vertices of degree 2 in C_4. Let $S_4^4(n,k)$ denote the graph obtained from $S_4^1(n-1,k)$ by attaching one pendant edge to the vertex f (see figure 2).

In this paper, we show that $S_4^1(n,k), S_4^2(n,k) (n \geqslant 2k)$ are the graphs with the largest and second largest spectral radius in $U^*(n,k)$, respectively, and $S_3^1(n,k)$ is the graph with the largest spread in $U(n,k)$.

2. Graphs with the largest and second largest spectral radius in $U^*(n,k)$

Lemma 1 [2]. Let uv be an edge of G, then

$$\phi(G,\lambda) = \phi(G-uv,\lambda) - \phi(G-u-v,\lambda) - 2\sum_{C\in\mathcal{C}(uv)} \phi(G-C,\lambda),$$

where $\mathcal{C}(uv)$ is the set of cycles that containing uv; In particular, if uv is a pendant edge with the pendant vertex v, then

$$\phi(G,\lambda) = \lambda\phi(G-v,\lambda) - \phi(G-u-v,\lambda).$$

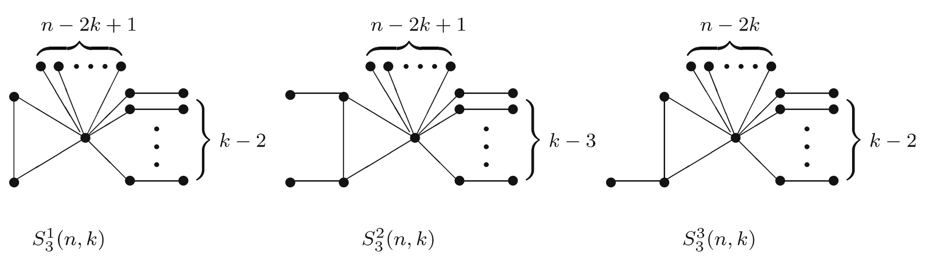

Figure 1. Graphs $S_3^1(n,k)$, $S_3^2(n,k)$ and $S_3^3(n,k)$.

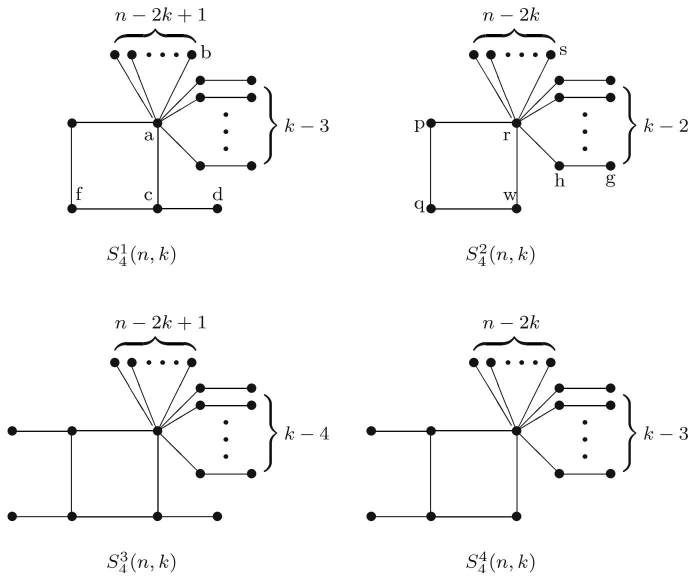

Figure 2. Graphs $S_4^i(n,k)$ for $i = 1, 2, 3, 4$.

Lemma 2 [10, 13]. Let G_1 and G_2 be two graphs. If $\phi(G_1,\lambda) < \phi(G_2,\lambda)$ for all $\lambda \geqslant \lambda_1(G_2)$, then $\lambda(G_1) > \lambda_1(G_2)$.

Lemma 3 [10]. Let G be a connected graph, and let G' be a proper spanning subgraph of G. Then

$$\phi(G',\lambda) > \phi(G,\lambda) \quad \text{for all } \lambda \geqslant \lambda_1(G).$$

Furthermore, we have $\lambda_1(G) > \lambda_1(G')$.

Unicyclic graphs are also viewed as planting some trees at vertices of the unique cycle of G. So, we can view the vertices $r_i (i = 1, \ldots, g)$ of $C_{g(G)}$ as roots, and T_i as planting tree at $r_i (r_i \in T_i)$.

Let $G \in U^*(2k,k)$. If $v \in V(T_i)$ is a vertex furthest from the root r_i, and the distance is less than 2, then v is a pendant vertex. Let u be the vertex adjacent to v. Then $d(u) = 2$. Otherwise, G has no perfect matching. We define a transformation (F): deleting the other edge that incident to u and adding an edge $r_i u$. Carry out transformation (F) to T_i repeatedly, we can obtain the graph G' such that only some paths of length 2 and at most one edge are attached to r_i.

Lemma 4 [1]. Let $G \in U^*(2k,k)$, G' be the graph as above. Then $G' \in U^*(2k,k)$ and

$$\phi(G,\lambda) > \phi(G',\lambda) \quad \text{for all } \lambda \geqslant \lambda_1(G).$$

In particular, $\lambda_1(G') > \lambda_1(G)$

If we apply transform (F) to all planting trees $T_i (i = 1, 2, \ldots, g(G))$ of G repeatedly, we can finally obtain a graph G'' such that for any vertex w of $C(G'')$, there are only some paths of length 2 and at most one pendant vertex that are attached to w.

Lemma 5 [7]. Let u and v be two vertices in a non-trivial connected graph G, and suppose that s paths of length 2 are attached to G at u, and t paths of length 2 are attached to G at v to form a graph $G_{s,t}$. Then either

$$\lambda_1(G_{s+i,t-i}) > \lambda_1(G_{s,t}) \ (1 \leqslant i \leqslant t) \quad or$$
$$\lambda_1(G_{s-i,t+i}) > \lambda_1(G_{s,t}) \ (1 \leqslant i \leqslant s).$$

Apply lemmas 4 and 5, we can get a graph H in $U(2k, k)$ such that all paths of length 2 are attached to one vertex of $C(H)$, other vertices are pendant ones, and just one of those is joining to one vertex of $C(H)$.

Lemma 6 [1]. Let $G_i, G_i' (i = 1, 2, 3)$ be the graphs shown in figure 3. Then

$$\phi(G_i, \lambda) > \phi(G_i', \lambda) \quad \text{for all } \lambda \geqslant \lambda(G_i)$$

In particular, we have $\lambda_1(G_i) < \lambda_1(G_i')$ for $i = 1, 2, 3$, respectively.

By lemmas 3 and 4, by a series of transforms, we can obtain a graph $G^* \in U^*(2k, k)$ such that $g(G^*) = 4$, a vertex of C_4 is attached by some paths of length 2 and at most one pendant edge, and other vertices of C_4 are attached by at most one pendant edge. Thus G^* must be a graph of figure 2.

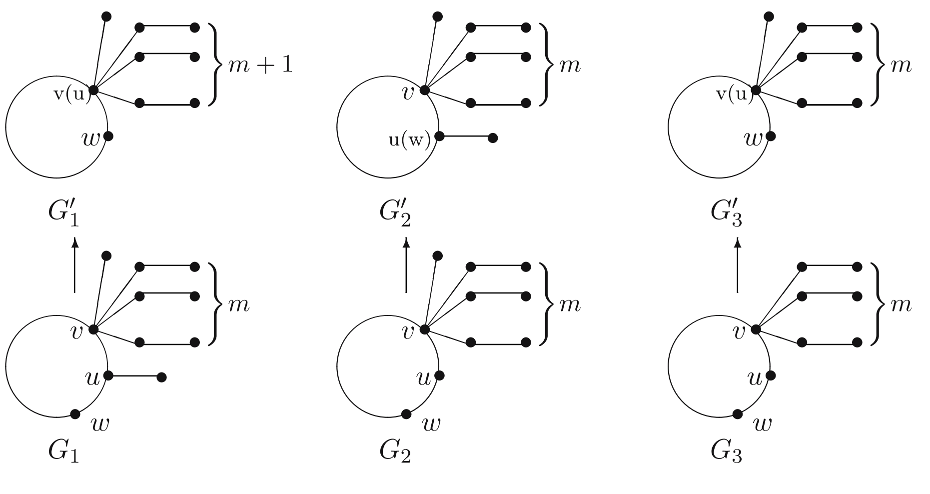

Figure 3. Graphs G_i and G_i' for $i = 1, 2, 3$.

Lemma 7.

$$\begin{aligned} &\phi(S_4^1(2k,k),\lambda) < \phi(S_4^2(2k,k),\lambda) \quad \text{for all } \lambda \geqslant \lambda(S_4^2(2k,k)),\\ &\phi(S_4^2(2k,k),\lambda) < \phi(S_4^3(2k,k),\lambda) \quad \text{for all } \lambda \geqslant \lambda(S_4^3(2k,k)),\\ &\phi(S_4^3(2k,k),\lambda) < \phi(S_4^4(2k,k),\lambda) \quad \text{for all } \lambda \geqslant \lambda(S_4^4(2k,k)). \end{aligned}$$

In particular, $\lambda_1(S_4^1(2k,k)) > \lambda_1(S_4^2(2k,k)) > \lambda_1(S_4^3(2k,k)) > \lambda_1(S_4^4(2k,k))$.

Proof. Let $e = fc, e' = pq$ as shown in figure 2. Delete e, e' from $S_4^1(2k,k)$, $S_4^2(2k,k)$, respectively. By lemma 1, we have

$$\begin{aligned} \phi(S_4^1(2k,k),\lambda) &= \phi(S_4^1(2k,k) - fc,\lambda) - \phi(S_4^1(2k,k) - f - c,\lambda)\\ &\quad -2\phi(2K_1 \cup (k-3)K_2,\lambda)\\ \phi(S_4^2(2k,k),\lambda) &= \phi(S_4^2(2k,k) - pq,\lambda) - \phi(S_4^2(2k,k) - p - q,\lambda)\\ &\quad -2\phi((k-2)K_2,\lambda) \end{aligned}$$

Obviously, $\phi(S_4^1(2k,k),\lambda) < \phi(S_4^2(2k,k),\lambda)$ for all $\lambda \geqslant \lambda_1(S_4^2(2k,k))$, since $S_4^1(2k,k) - f - c, 2K_1 \cup (k-3)K_2$ are subgraphs of $S_4^2(2k,k) - p - q, (k-2)K_2$, respectively, and $S_4^1(2k,k) - fc \cong S_4^2(2k,k) - pq$. By lemma 2, we have $\lambda(S_4^1(2k,k)) > \lambda_1(S_4^2(2k,k))$. Similarly, we can obtain

$$\begin{aligned} &\phi(S_4^2(2k,k),\lambda) < \phi(S_4^3(2k,k),\lambda) \quad \text{for all } \lambda \geqslant \lambda(S_4^3(2k,k))\\ &\phi(S_4^3(2k,k),\lambda) < \phi(S_4^4(2k,k),\lambda) \quad \text{for all } \lambda \geqslant \lambda(S_4^4(2k,k)). \end{aligned}$$

Furthermore, we have $\lambda_1(S_4^2(2k,k)) > \lambda_1(S_4^3(2k,k)) > \lambda_1(S_4^4(2k,k))$. □

In order to describe our results better, we first give the following lemma.

Lemma 8. Let $G \in U^*(2k,k), G \not\cong S_4^1(2k,k), S_4^2(2k,k), v \in V(C(G))$. If there exists a path $P = vv_1v_2$ of length 2 attached to v and $G - v_1 - v_2 \not\cong S_4^1(2k-2, k-1)$, then

$$\phi(G,\lambda) > \phi(S_4^2(2k,k),\lambda) \quad \text{for all } \lambda \geqslant \lambda_1(G).$$

Proof. By induction on k. By lemma 1, we have

$$\begin{aligned} \phi(G,\lambda) &= \phi(G - vv_1,\lambda) - \phi(G - v - v_1,\lambda)\\ \phi(S_4^2(2k,k),\lambda) &= (\lambda^2 - 1)\phi(S_4^2(2k-2,k-1),\lambda) - \phi(P_3 \cup P_1 \cup (k-3)P_2,\lambda) \end{aligned}$$

Let $G - vv_1 = G' \cup v_1v_2$. Then, $G' \in U^*(2(k-1),k-1)$, and $\phi(G,\lambda) = (\lambda^2 - 1)\phi(G',\lambda)$. By induction hypothesis, we have

$$(\lambda^2 - 1)\phi(S_4^2(2(k-1),k-1) \leqslant (\lambda^2 - 1)\phi(G',\lambda) \quad \text{for } \lambda \geqslant \lambda_1(G).$$

If there is no pendant vertex adjacent to v, then $P_3 \cup P_1 \cup (k-3)P_2$ is subgraph of $G - v - v_1$. Using the result above and lemmas 2 and 3, we can obtain the result.

If there exists a pendant vertex adjacent to v, then $P_4 \cup 2P_1 \cup (k-4)P_2$ is a subgraph of $G - v - v_1$. For $\lambda > \lambda_1(G)$,

$$\begin{aligned}&\phi(P_3 \cup P_1 \cup (k-3)P_2, \lambda) - \phi(P_4 \cup 2P_1 \cup (k-4)P_2, \lambda)\\&\quad = \lambda^2(\lambda^2-2)(\lambda^2-1)^{k-3} - \lambda^2(\lambda^4 - 3\lambda^2 + 1)(\lambda^2-1)^{k-4}\\&\quad = \lambda^2(\lambda^2-1)^{k-4} > 0.\end{aligned}$$

Similarly, the result follows. □

Lemma 9. Let $G \in U^*(2k,k)$, and $G \not\cong S_4^1(2k,k)$ *or* $S_4^2(2k,k)$. Then

$$\phi(S_4^2(2k,k), \lambda) < \phi(G, \lambda) \quad \text{for all } \lambda \geqslant \lambda_1(G).$$

In particular, $\lambda_1(S_4^2(2k,k)) > \lambda_1(G)$.

Proof. It is trivial for $k = 2$. From the tables of [3,4], we can obtain the result for $k = 3, 4$. Suppose, now $k \geqslant 5$. If G is finally transformed into one of the graphs $S_4^2(2k,k)$, $S_4^3(2k,k)$ or $S_4^4(2k,k)$, then by lemmas 5, 6, 7, and 8, the lemma holds. If G is transformed into $S_4^1(2k,k)$, let G' be transformed into $S_4^1(2k,k)$ at the last step, then $g(G') = 4$ or $g(G') = 5$.

If $g(G') = 5$, then G' satisfies the condition of lemma 8. By lemmas 5, 6, and 7, we can obtain the result.

If $g(G') = 4$, then either G' satisfies the condition of lemma 8 or G' is the graph obtained from $S_4^1(8,4)$ by attaching a path of length 2 to a vertex of planting subtree P_3 or a vertex of degree 3 in C_4. We can obtain the result by simple computation and lemmas 5,6,7, and 8. □

Applying lemmas 7 and 9, we can obtain

Lemma 10. Let $G \in U^*(2k,k)$, and $G \not\cong S_4^1(2k,k)$, then

$$\phi(S_4^1(2k,k), \lambda) < \phi(G, \lambda) \quad \text{for all } \lambda \geqslant \lambda_1(G).$$

In particular, $\lambda_1(S_4^1(2k,k)) > \lambda_1(G)$.

Lemma 11 [14]. Let $G \in U(n,k), G \not\cong C_n$ $(n > 2k)$. Then there is a maximal matching M and a pendant vertex v such that M does not meet v.

Theorem 12. $S_4^1(n,k)$ is the graph with the maximal spectral radius in $U^*(n,k)$.

Proof. By induction on n. The result holds for $n = 2k$ by lemma 7. Suppose, that it is true for $n \leqslant m - 1$. Let $n = m\,(m > 2k)$. There exists a pendant edge e that does not belong to a maximal matching M of G. Let $e = wr$ with pendant vertex r. By lemma 1, we have

$$\begin{aligned}\phi(S_4^1(n,k),\lambda) &= \lambda\phi(S_4^1(n-1,k),\lambda) - \phi(P_4 \cup (n-2k)K_1 \cup (k-3)K_2, \lambda),\\ \phi(G,\lambda) &= \lambda\phi(G-r,\lambda) - \phi(G-w-r,\lambda),\end{aligned}$$

where $G \in U^*(n,k), G \not\cong S_4^1(n,k), S_4^2(n,k)$. By induction hypothesis, we have $\phi(S_4^1(n-1,k),\lambda) < \phi(G-r,\lambda)$ for $\lambda > \lambda_1(G-r)$.

It suffices to prove $\phi(P_4 \cup (n-2k)K_1 \cup (k-3)K_2, \lambda) > \phi(G-w-r,\lambda)$ for $\lambda > \lambda_1(G)$. Since, any maximal matching M of G that misses r must meet w, otherwise $M \cup wr$ is a matching of G. $G-w-r$ has a maximal matching with value $k-1$.

Case 1. $P_4 \cup (n-2k)K_1 \cup (k-3)K_2$ is a subgraph of $G-w-r$. By lemma 3, the result holds.

Case 2. $P_4 \cup (n-2k)K_1 \cup (k-3)K_2$ is not a subgraph of $G-w-r$. Let M' be an maximal matching of $G-w-r$. Let V' be the vertex set of M', and $V'' = V(G-w-r) - V'$.

Claim 1. $G[V'']$ is empty. Otherwise, let $e_1 \in E(G[V''])$, then $M' \cup e_1$ is a matching of $G-w-r$ with cardinality k.

Claim 2. $G[V'] \setminus E(M')$ is empty.

Claim 3. $v \in V''$ is adjacent to at most one vertex of V' in $G-w-r$.

Claim 4. For any edge $e_2 = ij$ of M', if i is adjacent to some vertices of V'', then j must not be adjacent to any vertex of $G-w-r$ except for i.

Claim 5. $G-w-r$ contains no cycle.

Claim 6. $g(G) = 4$. If $g(G) \geqslant 5$, then $G-w-r$ can not satisfy the above claims.

So, the components of $G-w-r$ are isolated vertices, P_2, or stars.

Let qx be an pendant edge of G with pendant vertex x. If q is not a vertex on the cycle, but q is adjacent to another pendant vertex in G, we can choose qx as deleting edge and x as deleting vertex. We know that $G-q-x$ contains a cycle. Then $P_4 \cup (n-2k)K_1 \cup (k-3)K_2$ must be a subgraph of $G-q-x$. Thus the result hold.

If G contains no such construction, then G must be a graph G'' as shown in figure 4. Since, $P_3 \cup (n-2k-1)K_1 \cup (k-2)K_2$ is a subgraph of $G''-r-z$

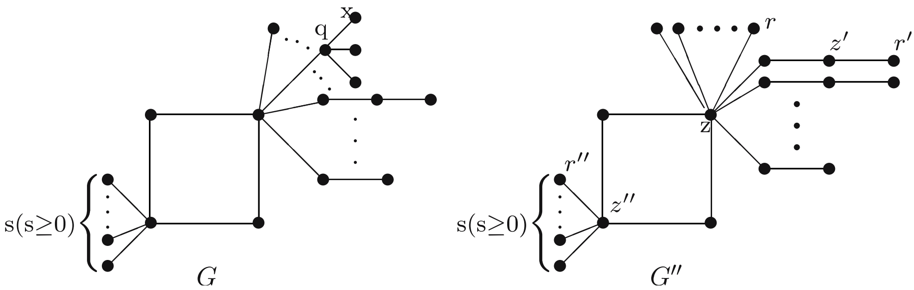

Figure 4. Graphs G and G''.

(or $G'' - r' - z'$ or $G'' - r'' - z''$), we can easily obtain that $\lambda_1(S_4^2(n,k)) > \lambda_1(G'')$ by induction.

In the following, we prove that $\lambda_1(S_4^1(n,k)) > \lambda_1(S_4^2(n,k))$.

Let c, d be vertices of $S_4^1(n,k)$ and g, h be vertices of $S_4^2(n,k)$ as shown in figure 2. By lemma 1, we have

$$\phi(S_4^1(n,k),\lambda) = \lambda\phi(S_4^1(n,k) - d, \lambda) - \phi(S_4^1(n,k) - c - d, \lambda),$$
$$\phi(S_4^2(n,k),\lambda) = \lambda\phi(S_4^2(n,k) - g, \lambda) - \phi(S_4^2(n,k) - h - g, \lambda).$$

It is obvious that $S_4^1(n,k) - d \cong S_4^2(n,k) - g$, and that $S_4^1(n,k) - c - d$ is a subgraph of $S_4^2(n,k) - g - h$. By lemmas 3 and 4, we can obtain the result. □

Theorem 13. $S_4^2(n,k)$ is the graph with the second maximal spectral radius in $U^*(n,k)$.

Proof. By induction on n. Let v be the pendant vertex of G not met by a maximal matching M, u be its adjacent vertex. By lemma 3, we have

$$\phi(S_4^2(n,k),\lambda) = \lambda\phi(S_4^2(n-1,k),\lambda) - \phi((n-2k-1)K_1 \cup P_3 \cup (k-2)K_2, \lambda),$$
$$\phi(G,\lambda) \qquad = \lambda\phi(G - v, \lambda) - \phi(G - u - v, \lambda),$$

where $G \in U^*(n,k)$ and $G \not\cong S_4^1(n,k), S_4^2(n,k)$.

Case 1. $P_3 \cup (n-2k-1)K_1 \cup (k-2)K_2$ is a subgraph of $G - u - v$. By Lemma 3, the result holds.

Case 2. $P_3 \cup (n-2k-1)K_1 \cup (k-2)K_2$ is not a subgraph of $G - u - v$. M', V', V'' are defined as in theorem 11. It is obvious that $E[V' : V'']$ and $E(G[V''])$ are empty. We also know that $E(G[V']) \setminus E(M)$ is not empty, otherwise, we can not reconstruction G such that it contains a cycle. So, $P_4 \cup (n-2k)K_1 \cup (k-3)K_2$ is a subgraph of $G - u - v$.

By lemma 3, for $\lambda > \lambda_1(G) > \lambda_1(G-u-v)$,

$$\phi(P_4 \cup (n-2k)K_1 \cup (k-3)K_2, \lambda) > \phi(G-u-v, \lambda)$$

and since for $\lambda > \lambda_1(G)$

$$\begin{aligned}&\phi(P_3 \cup (n-2k-1)K_1 \cup (k-2)K_2, \lambda) - \phi(P_4 \cup (n-2k)K_1 \cup (k-3)K_2, k), \lambda)\\ &= \lambda^{n-2k}(\lambda^2-2)(\lambda^2-1)^{k-2} - \lambda^{n-2k}(\lambda^4-3\lambda^2+1)(\lambda^2-1)^{k-3}\\ &= \lambda^{n-2k}(\lambda^2-1)^{k-3} > 0\end{aligned}$$

combining the induction hypothesis and lemmas 3 and 5, we can obtain the result. □

3. The graph with maximal spread in U(n,k)

In order to describe our results, we need give some definitions and lemmas. Let $T(n,k)$ be the set of all trees on n vertices with a maximal matching of cardinality k. Let $A(n,k)$, $B(n,k)$, $C(n,k)$ be the trees as shown in figure 5.

Lemma 14 [8]. $A(n,k)$, $B(n,k)$ $(n > 2k)$ are the graphs with the maximal and second maximal spectral radius in $T(n,k)$, respectively; $A(2k,k)$, $C(2k,k)$ are the graphs with the maximal and second maximal spectral radius in $T(2k,k)$, respectively.

Lemma 15 [5]. Let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G) (\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G))$ be the eigenvalues of graph G. If G is connected, then

$$|\lambda_i(G)| \leqslant \lambda_1(G), \quad i = 1, 2, \ldots, n.$$

If G is bipartite, then $\lambda_1(G) = -\lambda_n(G)$.

Lemma 16 [14]. $S_3^1(n,k)$ is the graph with the largest spectral radius in $U(n,k)$ except for $n = 2k = 6$.

Lemma 17. Let $G \in U(n,k)$, $g(G) = 3$. Then there exists an edge e of $E(C_3)$ such that $\lambda_n(G-e) \leqslant \lambda_n(G)$.

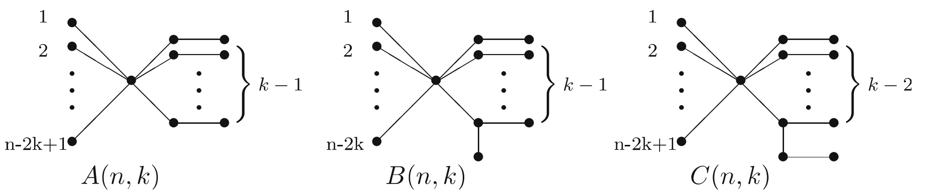

Figure 5. Graphs $A(n,k)$, $B(n,k)$, and $C(n,k)$.

Proof. Let $X = (x_1, x_2, x_3, \ldots, x_n)^T$ be the unity eigenvector of $\lambda_n(G)$, where $C_3(G) = v_1v_2v_3$ and x_1, x_2, x_3 correspond to $v_1v_2v_3$, respectively. Then there exist i, j $(1 \leqslant i < j \leqslant 3)$ such that $x_ix_j \geqslant 0$. Otherwise, $x_1x_2 < 0, x_2x_3 < 0, x_3x_1 < 0$, which is impossible. By Rayleigh quotient, we have

$$\begin{aligned}\lambda_n(G - v_iv_j) &\leqslant X^TA(G - v_iv_j)X = X^TA(G)X - 2x_ix_j \\ &= \lambda_n(G) - 2x_ix_j \leqslant \lambda_n(G).\end{aligned}$$

□

Lemma 18. Let $G \in U(n,k), g(G) = 3$, and $G \not\cong S_3^1(n,k)$. Then $\lambda_n(S_4^1(n,k)) \leqslant \lambda_n(G)$ for $k \geqslant 3$; $\lambda_n(S_4^2(n,k)) \leqslant \lambda_n(G)$ for $k = 2$.

Proof. By lemma 4, we can obtain a tree G' from G by deleting an edge of C_3 such that $\lambda_n(G') \leqslant \lambda_n(G)$.

If $G' \in T(n,k)$, by lemma 3, we have $\lambda_1(A(n,k)) \geqslant \lambda_1(G')$. Since, $A(n,k)$ is a subgraph of $S_4^1(n,k)$, by lemma 3, we have $\lambda_1(S_4^1(n,k) \geqslant \lambda_1(A(n,k))$. Since, $S_4^1(n,k), G'$ are all bipartite, by lemma 15, we have

$$\lambda_n(S_4^1(n,k)) = -\lambda_1(S_4^1(n,k)) \leqslant -\lambda_1(A(n,k)) \leqslant -\lambda_1(G') = \lambda_n(G') \leqslant \lambda_n(G).$$

If $G' \in T(n,k-1)$, since $G \not\cong S_3^1(n,k)$ we know $G' \not\cong A(n,k-1)$. Since, $B(n,k-1)$ is a subgraph of $S_4^1(n,k)$, similarly, we can obtain $\lambda_n(S_4^1(n,k)) \leqslant \lambda_n(G)$.

If $k = 2$, we can obtain that $\lambda_1(S_4^2(n,k)) > \lambda_1(B(n,k-1)$ easily. Similar to the above proof, we can obtain the result. □

Lemma 19. $\lambda_n(S_3^1(n,k)) < \lambda_n(S_4^1(n,k))$ $(k \geqslant 3)$, for $n \geqslant 18$; $\lambda_n(S_3^1(n,k)) < \lambda_n(S_4^2(n,k))$ $(k = 2)$, for $n \geqslant 12$.

Proof. By lemma 3, we can get

$$\begin{aligned}\phi(S_3^1(n,k),\lambda) &= \lambda^{n-2k}(\lambda^2-1)^{k-2}[\lambda^4 - (n-k+2)\lambda^2 - 2\lambda + (n-2k+1)] \\ \phi(S_4^2(n,k),\lambda) &= \lambda^{n-2k}(\lambda^2-1)^{k-3}[\lambda^6 - (n-k+3)\lambda^4 + (3n-4k)\lambda^2 - (2n-4k)] \\ \phi(S_4^1(n,k),\lambda) &= \lambda^{n-2k}(\lambda^2-1)^{k-4}[\lambda^8 - (n-k+4)\lambda^6 + (4n-5k+2)\lambda^4 \\ &\quad -(4n-7k+3)\lambda^2 + n - 2k + 1].\end{aligned}$$

Let

$$\begin{aligned}f(x) &= x^4 - (n-k+4)x^3 + (4n-5k+2)x^2 - (4n-7k+3)x + n - 2k + 1 \\ h(\lambda) &= \lambda^6 - (n-k+3)\lambda^4 + (3n-4k)\lambda^2 - (2n-4k) \\ g(\lambda) &= \lambda^4 - (n-k+2)\lambda^2 - 2\lambda + (n-2k+1).\end{aligned}$$

We know that $f(0) = n - 2k + 1 > 0$, $f(\frac{3-\sqrt{5}}{2}) = -x^3 < 0 (x > 0)$, $f(1) = k - 3 > 0$, $f(\frac{3+\sqrt{5}}{2}) = -x^3 < 0 (x > 0)$, $g(-\sqrt{n-k+2}) = 2\sqrt{n-k+2} + (n - 2k + 1) > 0$. Let $n = 2k + m$ $(k \geqslant 3)$.

If $0 \leqslant m \leqslant k$, then

$$g(-\sqrt{n-k+\frac{6}{5}}) = -\frac{1}{5}(4k - m - \frac{1}{5} - 10\sqrt{k+m+\frac{6}{5}}) < 0$$
$$(m \leqslant k\ k \geqslant 24, m \leqslant k-1\ k \geqslant 22, m \leqslant k-2\ k \geqslant 22,$$
$$m \leqslant k-3\ k \geqslant 20, m \leqslant k-4 k \geqslant 19, m \leqslant k-5\ k \geqslant 17,$$
$$m \leqslant k-6\ k \geqslant 16, m \leqslant k-7\ k \geqslant 13, m \leqslant k-8\ k \geqslant 8)$$
$$f(n-k+\frac{6}{5}) = \frac{1}{5}[(k + 6m - \frac{44}{5})(5k + 5m - 9) - 195] > 0$$
$$(m \geqslant 0\ k \geqslant 13, m \geqslant 1\ k \geqslant 8, m \geq 20\ k \geqslant 5, m \geqslant 3\ k \geqslant 3).$$

Combining the Appendix table, We can obtain that $\lambda_n(S_3^1(n,k)) < \sqrt{n-k+\frac{6}{5}} < \lambda_n(S_4^1(n,k))$, $(n \geqslant 18)$.

If $m \geqslant k + 1$, then

$$g(-\sqrt{n-k+\frac{2}{3}}) = -\frac{1}{3}(\sqrt{k+m+\frac{2}{3}} - 3)^2 - k + 3 + \frac{1}{3} < 0 (k \geqslant 4, k = 3\ m \geqslant 1)$$
$$f(n-k+\frac{2}{3}) = \frac{1}{3}(k+m+\frac{2}{3})[(3k+3m-7)$$
$$(-k+2m-\frac{20}{3}) - 75] + m + 1 > 0$$
$$(m \geqslant k+1\ k \geqslant 7, m \geqslant k+2\ k \geqslant 6, m \geq k+3\ k \geqslant 4,$$
$$m \geqslant k+5\ k \geqslant 3).$$

Combining the Appendix table, we can obtain that $\lambda_n(S_3^1(n,k)) < \sqrt{n-k+\frac{2}{3}} < \lambda_n(S_4^1(n,k))$, $(n \geqslant 18)$.

If $k = 2$. $h(x) = x^6 - (m+5)x^4 + (3m+4)x^2 - 2m = (x^2 - 1)[x^4 - (m+4)x^2 + 2m]$. When $m \geqslant 8$,

$$g(2+m+\frac{1}{2}) = -\frac{1}{2}(\sqrt{2+m+\frac{1}{2}} - 2)^2 + \frac{1}{2} < 0$$
$$\lambda_n(S_4^2(n,2)) = -\sqrt{\frac{(m+4)+\sqrt{m^2+16}}{2}} > -\sqrt{m+\frac{5}{2}}.$$

Thus, we have $\lambda_n(S_3^1(n,2)) < -\sqrt{m+\frac{5}{2}} < \lambda_n(S_4^2(n,2))$. □

Appendix table

$n = 2k + m$	$\lambda_n(S_4^1(n,k))$	$\lambda_n(S_3^1(n,k))$
$k = 6$ $m = 8\ 7\ 6\ 5$	3.8256 3.6997 3.5705 3.4380	3.8570 3.7266 3.5917 3.4521
$k = 7$ $m = 7\ 6\ 5\ 4$	3.8347 3.7097 3.5815 3.4563	3.8662 3.7368 3.6032 3.4650
$m = 3$	3.3160	3.3217
$k = 8$ $m = 8\ 7\ 6\ 5$	4.0834 3.9649 3.8437 3.7195	4.1213 4.0000 3.8753 3.7468
$m = 4\ 3\ 2\ 1$	3.5923 3.4623 3.3295 3.1940	3.6144 3.4777 3.3363 3.1901
$k = 9$ $m = 9\ 8\ 7\ 6\ 5$	4.3187 4.2060 4.0909 3.9730 3.8525	4.3607 4.2462 4.1287 4.0082 3.8843
$m = 4\ 3\ 2\ 1$	3.7292 3.6031 3.4742 3.3426	3.7568 3.6255 3.4901 3.3504
$k = 10$ $m = 10\ 9\ 8\ 7\ 6$	4.7560 4.6529 4.5479 4.4407 4.3313	4.5871 4.4783 4.3669 4.2529 4.1361
$m = 5\ 4\ 3\ 2$	4.2196 4.1055 3.9890 3.8694	4.6162 3.8931 3.7666 3.6364
$k = 11$ $m = 11\ 10\ 9\ 8\ 7$	4.7560 4.6529 4.5479 4.4407 4.3313	4.8024 4.6985 4.5924 4.4840 4.3731
$m = 6\ 5\ 4\ 3$	4.2196 4.1055 3.9890 3.8694	4.2597 4.1434 4.0242 3.9019
$k = 12$ $m = 12\ 11\ 10\ 9\ 8$	4.9607 4.8616 4.7607 4.6580 4.5532	5.0083 5.9086 4.8071 4.7035 4.5977
$m = 7\ 6\ 5\ 4$	4.4464 4.3375 4.2263 4.1128	4.4897 4.3793 4.2663 4.1507
$k = 13$ $m = 15\ 14\ 13\ 12$	5.1574 5.0619 4.9648 4.8660	5.2057 5.1099 5.0123 4.9130
$m = 11\ 10\ 9\ 8$	4.7634 4.6630 4.5586 4.4522	4.8117 4.7084 4.6030 4.4954
$k = 14$ $m = 14\ 13\ 12\ 11$	5.3471 5.2548 5.1611 5.0658	5.3958 5.3033 5.2093 5.1137
$m = 10\ 9\ 8\ 7$	4.9690 4.8704 4.7701 4.6679	5.0164 4.9173 4.8163 4.7133
$k = 15$ $m = 15\ 14\ 13\ 12$	5.5303 5.4410 5.3504 5.2583	5.5792 5.4898 5.3990 5.3067
$m = 11\ 10\ 9\ 8$	5.1648 5.0697 4.9731 4.8748	5.2129 5.1175 5.0204 4.9216
$k = 16$ $m = 16\ 15\ 14\ 13$	5.7078 5.6211 5.5333 5.4442	5.7567 5.6700 5.8214 5.4929
$m = 12\ 11\ 10$	5.3337 5.2618 5.1685	5.4023 5.3102 5.2165
$k = 17$ $m = 17\ 16\ 15\ 14$	5.8799 5.7958 5.7105 5.6240	5.9288 5.8447 5.7594 5.6729
$m = 13\ 12$	5.5363 5.4473	5.5851 5.4960
$k = 18$ $m = 18\ 17\ 16\ 15$	6.0472 5.9653 5.8824 5.7984	6.0960 6.0141 5.9312 5.8472
$m = 14\ 13$	5.7132 5.6268	5.7620 5.6757
$k = 19$ $m = 19\ 18\ 17\ 16\ 15$	6.2169 6.1303 6.0495 5.9677 5.8849	6.2586 6.1789 6.0982 6.0164 5.9337
$k = 20$ $m = 20\ 19\ 18\ 17\ 16$	6.3688 6.2909 6.2122 6.1325 6.0517	6.4170 6.3393 6.2607 6.1810 6.1004
$k = 21$ $m = 21\ 20\ 19\ 18$	6.524 6.4476 6.3707 6.2930	6.5716 6.4957 6.4190 6.3413
$k = 22$ $m = 22\ 21\ 20$	6.6750 6.6006 6.5255	6.7226 6.6484 6.5734
$k = 23$ $m = 23$	6.8230	6.8703

Theorem 20. $S_3^1(n,k)$ $(n \geqslant 18$ $k \geqslant 2)$ is the graph with the largest spread in $U(n,k)$.

Proof. Let $G \in U(n,k)$. If $g(G) = 3$, by lemmas 16 and 18, we can obtain $s(S_3^1(n,k)) > s(G)$. If $g(G) \geqslant 4$, by lemmas 12, 15, 16, and 19, we can obtain $s(S_3^1(n,k)) > s(S_4^1(n,k)) \geqslant s(G)$ for $k \geqslant 3$ and $s(S_3^1(n,k)) > s(S_4^2(n,k)) \geqslant s(G)$ for $k = 2$. □

Remark. Theorem 20 is still true except for a few graphs (see the Appendix table) with $n \leqslant 17$, for example, $S_4^1(15,5)$. For convenience, we just consider the case for $n \geqslant 18$.

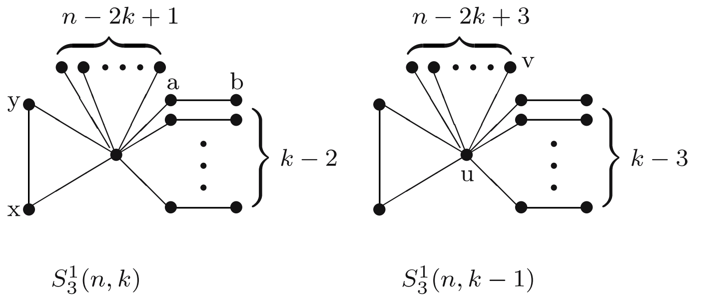

Figure 6. Graphs $S_3^1(n,k)$, $S_3^1(n,k-1)$ and vertices a, b, u, v.

Lemma 21. $s(S_3^1(n,k)) < s(S_3^1(n,k-1))$ $(n \geqslant 18$ $k \geqslant 3)$.

Proof. We first prove that $\lambda_1(S_3^1(n,k)) < \lambda_1(S_3^1(n,k-1))$. We can delete the pendant vertices v, b of $S_3^1(n,k-1)$, $S_3^1(n,k)$, respectively (see figure 6).

Similar to the proof of lemma 10, we can obtain the result. Since, x, y are symmetrical, by lemma 17, we have $\lambda_n(S_3^1(n,k) - xy) \leqslant \lambda_n(S_3^1(n,k))$. Since $\lambda_n(S_3^1(n,k) - xy)$ is a subgraph of $S_4^2(n,k-1)$, then $\lambda_n(S_3^1(n,k) - xy) > \lambda_n(S_4^2(n,k-1))$. By lemmas 11 and 19, we have $\lambda_n(S_3^1(n,k-1)) < \lambda_n(S_3^1(n,k)$. Thus $s(S_3^1(n,k)) < s(S_3^1(n,k-1))$ $(k \geqslant 3)$. □

Using theorem 20 and lemma 21, it is not difficult to obtain the following theorem.

Theorem 22. $S_3^1(n,2)$ $(n \geqslant 18)$ is the unique graph with the largest spread in the class of all unicyclic graphs with n vertices.

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